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Path integral for a three body problem

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Abstract. A one dimensional three body problem is treated by the path integral approach of Feynman. When expressed in suitable coordinates, it is possible to extract the symmetry from the path integral of the problem. The resulting propagator is evaluated in a closed analytical form.

1. Introduction

The path integral formulation of Feynman (1948) offers an alternative way of solving dynamical problems in quantum mechanics. Instead of the usual Schrödinger equation, this formulation considers the integral equation

$$\psi(\boldsymbol{r}'', t) = \int K(\boldsymbol{r}'', \boldsymbol{r}', t)\psi(\boldsymbol{r}', 0)\, \mathrm{d}\boldsymbol{r}' \tag{1}$$

with the initial condition $\psi(\boldsymbol{r}'', 0) = \psi(\boldsymbol{r}', 0)$; the propagator or the kernel K is defined by a path integral

$$K(\boldsymbol{r}'', \boldsymbol{r}', t) = \int \exp\{\mathrm{i}S(\boldsymbol{r}'', \boldsymbol{r}')\}\mathscr{D}\boldsymbol{r}(t). \tag{2}$$

Here, the integrations are over all possible paths, or histories, starting at $\boldsymbol{r}' = \boldsymbol{r}(0)$ and terminating at $\boldsymbol{r}'' = \boldsymbol{r}(t)$. The function $S(\boldsymbol{r}'', \boldsymbol{r}')$ in the integrand is the classical action

$$S(\boldsymbol{r}'', \boldsymbol{r}') = \int_0^t L(\dot{\boldsymbol{r}}, \boldsymbol{r})\, \mathrm{d}t \tag{3}$$

$L(\dot{\boldsymbol{r}}, \boldsymbol{r})$ being the Lagrangian of the system considered.

Although this approach has a great deal of intuitive appeal, its applicability is limited because of the difficulty of evaluating the path integrals. Explicit expressions for the path integrals are available only for a few cases (Feynman and Hibbs 1965). Further, in most of the applications, calculations are done in cartesian coordinates. Path integrals in polar coordinates were first considered by Edwards and Gulyaev (1964). The relevance of using polar coordinates is apparent when one considers, for example, the centrally symmetric potentials (Peak and Inomata 1969).

In the present paper, we present an example where path integration in polars is exploited to solve a problem with a potential of a symmetry more general than central. This situation arises in connection with a three body problem considered by Calogero (1969). This problem involves three equal mass particles in one dimension, with equal strength harmonic forces between every two particles and an additional interaction which varies as the inverse square of the interparticle distance between any one pair of particles. The example is interesting from the path integral point of view because the form of the inverse square potential (centrifugal potential) is responsible for a natural separation of the angular part (symmetry) from the propagator. The radial part can subsequently be evaluated analytically in the closed form. Furthermore, it is also possible to obtain an expansion of the propagator in terms of the eigenfunctions of the Schrödinger equation corresponding to the problem. Throughout this paper we use units such that $\hbar = m = 1$, where m is the mass of the particles.

2. Path integral for the three body problem

The problem under consideration is characterized by the Lagrangian

$$\begin{aligned} L &= \tfrac{1}{2}\{(\dot{\xi}^1)^2+(\dot{\xi}^2)^2+(\dot{\xi}^3)^2\} \\ &\quad -\tfrac{1}{4}\omega^2\{(\xi^1-\xi^2)^2+(\xi^2-\xi^3)^2+(\xi^3-\xi^1)^2\}-g(\xi^1-\xi^2)^{-2} \end{aligned} \tag{4}$$

where ξ^1, ξ^2, ξ^3 are the coordinates of the three particles, ω the angular frequency arising from the strength of the harmonic potentials and g is the strength of the inverse square potential acting between particles 1 and 2. We assume $g > -\frac{1}{4}$ to avoid the two body collapse (Landau and Lifshitz 1958).

In order to obtain the path integral, we use the usual definition

$$K(\boldsymbol{\xi}'', \boldsymbol{\xi}'; t) = \lim_{N\to\infty} A_N \int \exp\left(\mathrm{i} \sum_{j=1}^{N} S(\boldsymbol{\xi}_j, \boldsymbol{\xi}_{j-1})\right) \prod_{j=1}^{N-1} \mathrm{d}\boldsymbol{\xi}_j \tag{5}$$

where $\boldsymbol{\xi}$ stands for the triplet (ξ^1, ξ^2, ξ^3), $\boldsymbol{\xi}_j = \boldsymbol{\xi}(t_j)$, $\boldsymbol{\xi}_0 = \boldsymbol{\xi}'$, $\boldsymbol{\xi}_N = \boldsymbol{\xi}''$, $t_j - t_{j-1} = t/N = \epsilon$ and A_N is the normalization factor in the Nth approximation. The action $S(\boldsymbol{\xi}_j, \boldsymbol{\xi}_{j-1})$ over a small time interval $t_j - t_{j-1} = \epsilon$ may be approximated by

$$\begin{aligned} S(\boldsymbol{\xi}_j, \boldsymbol{\xi}_{j-1}) &= \epsilon L(\boldsymbol{\xi}_j, \boldsymbol{\xi}_{j-1}, \epsilon) \\ &= \frac{1}{2\epsilon}\sum_{i=1}^{3}(\xi^i_j - \xi^i_{j-1})^2 - \frac{\epsilon\omega^2}{4}\{(\xi^1_j-\xi^2_j)^2+(\xi^2_j-\xi^3_j)^2+(\xi^3_j-\xi^1_j)^2\} \\ &\qquad - \frac{\epsilon g}{(\xi^1_j-\xi^2_j)(\xi^1_{j-1}-\xi^2_{j-1})}. \end{aligned} \tag{6}$$

If we now use the centre of mass (CM) and Jacobi coordinates

$$\begin{aligned} \xi^1+\xi^2+\xi^3 &= 3R \\ \xi^1-\xi^2 &= \sqrt{2}x \\ \xi^1+\xi^2-2\xi^3 &= \sqrt{6}y \end{aligned} \tag{7}$$

the action $S(\boldsymbol{\xi}_j, \boldsymbol{\xi}_{j-1})$ may be written as

$$S(\boldsymbol{\xi}_j, \boldsymbol{\xi}_{j-1}) = S_0(R_j, R_{j-1}) + \bar{S}(x_j, y_j; x_{j-1}, y_{j-1}) \tag{8}$$

where

$$S_0(R_j, R_{j-1}) = \frac{3}{2\epsilon}(R_j - R_{j-1})^2$$

$$\bar{S}(x_j, y_j; x_{j-1}, y_{j-1}) = \frac{1}{2\epsilon}\{(x_j - x_{j-1})^2 + (y_j - y_{j-1})^2\} - \frac{3\omega^2\epsilon}{4}(x_j^2 + y_j^2) - \frac{g\epsilon}{2x_j x_{j-1}}.$$

The separation of the action S into S_0 and $\bar{S}$ allows us to write

$$K(\xi'', \xi', t) = K_0(R'', R', t)\bar{K}(x'', y''; x', y', t). \tag{9}$$

Here K_0 is a free particle propagator corresponding to the motion of the centre of mass. $\bar{K}$ represents the relative motion and is given by

$$\bar{K}(x'', y''; x', y', t) = \lim_{N\to\infty} B_N \int \exp\left(\mathrm{i}\sum_{j=1}^{N} \bar{S}(x_j, y_j; x_{j-1}, y_{j-1})\right) \prod_{j=1}^{N-1} \mathrm{d}x_j\,\mathrm{d}y_j \tag{10}$$

where B_N is the new normalization constant.

The natural symmetry involved in K can be seen by going over to the 'plane polar' coordinates (r, θ)

$$r = (x^2 + y^2)^{1/2} \quad (0 \leqslant r < \infty) \qquad \theta = \tan^{-1}\frac{x}{y} \quad (0 \leqslant \theta < 2\pi)$$

$$x = r\sin\theta \qquad y = r\cos\theta. \tag{11}$$

In fact using equations (7) and (11) the Lagrangian L of equation (4) takes the form

$$L = L_0 + \bar{L} \tag{12a}$$

where

$$L_0 = \tfrac{3}{2}\dot{R}^2 \tag{12b}$$

$$\bar{L} = \tfrac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \tfrac{3}{4}\omega^2 r^2 - \frac{g}{2r^2\sin^2\theta}. \tag{12c}$$

Indeed our aim is to obtain the path integral for a system with a Lagrangian given by equation (12). As discussed above, L_0 which corresponds to CM motion yields the free particle propagator K_0. The motion relative to the CM is described by $\bar{L}$ and is our main concern here. Classically $\bar{L}$ describes the motion of a single particle in a plane under a potential which contains a harmonic term and a term depending on both r and θ. This dependence of potential on r and θ makes this problem new and more interesting than the ones that have been hitherto path integrated. With these remarks and dropping the bar on K and S we obtain

$$S(x_j, y_j; x_{j-1}, y_{j-1}) = S(r_j, \theta_j; r_{j-1}, \theta_{j-1})$$
$$= \frac{1}{2\epsilon}\{r_j^2 + r_{j-1}^2 - 2r_j r_{j-1}\cos(\theta_j - \theta_{j-1})\} - \epsilon V(r_j) - \frac{\epsilon(a^2 - \frac{1}{4})}{2r_j r_{j-1}\sin\theta_j\sin\theta_{j-1}}$$

and the integrand of equation (10) becomes

$$\exp\left(\mathrm{i}\sum_{j=1}^{N} S(x_j, y_j; x_{j-1}, y_{j-1})\right)$$

$$= \prod_{j=1}^{N} \exp\left(\frac{\mathrm{i}}{2\epsilon}(r_j^2 + r_{j-1}^2) - i\epsilon V(r_j)\right) \exp\left(\frac{r_j r_{j-1}}{\mathrm{i}\epsilon}\cos\theta_j \cos\theta_{j-1}\right)$$

$$\times \exp\left(\frac{r_j r_{j-1}}{\mathrm{i}\epsilon}\sin\theta_j \sin\theta_{j-1} - \frac{\mathrm{i}\epsilon(a^2 - \frac{1}{4})}{2r_j r_{j-1}\sin\theta_j \sin\theta_{j-1}}\right) \tag{13}$$

where

$$a = \tfrac{1}{2}(1 + 4g)^{1/2} \tag{14a}$$

$$V(r_j) = \tfrac{3}{4}\omega^2 r_j^2. \tag{14b}$$

Noting that the asymptotic form of $I_a(u/\epsilon)$, the modified Bessel function, for small ϵ, is given by

$$I_a\left(\frac{u}{\epsilon}\right) \sim \left(\frac{\epsilon}{2\pi u}\right)^{1/2} \exp\left(\frac{u}{\epsilon} - \frac{1}{2}\left(a^2 - \frac{1}{4}\right)\frac{\epsilon}{u} + \mathrm{O}(\epsilon^2)\right) \tag{15}$$

we may replace the last exponential in equation (13) by

$$\left(\frac{2\pi u}{\epsilon}\right)^{1/2} I_a\left(\frac{u}{\epsilon}\right) \tag{16}$$

with

$$u = \frac{r_j r_{j-1}}{\mathrm{i}}\sin\theta_j \sin\theta_{j-1}. \tag{17}$$

Use of equation (16) and the expansion formula (Erdélyi 1953 p102)

$$(\sin\alpha\sin\beta)^{1/2-\lambda} I_{\lambda-1/2}(z\sin\alpha\sin\beta)\exp(z\cos\alpha\cos\beta)$$

$$= 2^{2\lambda}(2\pi z)^{-1/2}(\Gamma(\lambda))^2 \sum_{l=0}^{\infty} \frac{l!(\lambda+l)}{\Gamma(2\lambda+l)} I_{l+\lambda}(z) C_l^{\lambda}(\cos\alpha) C_l^{\lambda}(\cos\beta) \tag{18}$$

enables us to write equation (13) as

$$\exp\left(\mathrm{i}\sum_{j=1}^{N} S(x_j, y_j; x_{j-1}, y_{j-1})\right) = \sum_{l_1, l_2, \ldots, l_N} \left(\prod_{j=1}^{N} N_{l_j}^2 (\sin\theta_j \sin\theta_{j-1})^{a+1/2} C_{l_j}^{a+1/2}(\cos\theta_j)\right.$$

$$\left.\times C_{l_j}^{a+1/2}(\cos\theta_{j-1}) R_{l_j}(r_j, r_{j-1})\right) \tag{19}$$

where N_l is the normalization factor of the Gegenbauer polynomial $C_l^{a+1/2}(\cos\theta)$ (Erdélyi 1953 p174) and

$$R_l(r_j, r_{j-1}) = (2\pi)\exp\left(\frac{\mathrm{i}}{2\epsilon}(r_j^2 + r_{j-1}^2) - \mathrm{i}\epsilon V(r_j)\right) I_{l+a+1/2}\left(\frac{r_j r_{j-1}}{\mathrm{i}\epsilon}\right). \tag{20}$$

Returning to equation (10), we may write

$$K(r'',\theta'';r',\theta',t) = \lim_{N\to\infty} B_N \sum_{l_1,l_2,\ldots,l_N} \int \prod_{j=1}^{N} (N_{l_j}^2(\sin\theta_j \sin\theta_{j-1})^{a+1/2} C_{l_j}^{a+1/2}(\cos\theta_j)$$
$$\times C_{l_j}^{a+1/2}(\cos\theta_{j-1}) R_{l_j}(r_j, r_{j-1})) \prod_{j=1}^{N-1} r_j \, \mathrm{d}r_j \, \mathrm{d}\theta_j. \tag{21}$$

The angular integrations in equation (21) can easily be performed by using the orthogonality relation

$$\int_0^\pi (\sin\theta)^{2a+1} C_l^{a+1/2}(\cos\theta) C_{l'}^{a+1/2}(\cos\theta)\,\mathrm{d}\theta = \frac{\delta_{ll'}}{N_l^2} \tag{22}$$

and the integral in equation (21) reduces to

$$\prod_{j=1}^{N-1} \delta_{l_j l_{j+1}} N_{l_N}^2 (\sin\theta'' \sin\theta')^{a+1/2} C_{l_1}^{a+1/2}(\cos\theta') C_{l_N}^{a+1/2}(\cos\theta''). \tag{23}$$

When this expression is substituted in equation (21), we see that for each quantum number l, the angular and radial contributions to the propagator are separable. Thus

$$K(r'',\theta'';r',\theta',t) = \sum_{l=0}^{\infty} K_l(r'',r',t) N_l^2 (\sin\theta'' \sin\theta')^{a+1/2} C_l^{a+1/2}(\cos\theta'') C_l^{a+1/2}(\cos\theta') \tag{24}$$

where

$$K_l(r'',r',t) = \lim_{N\to\infty} B_N \int \prod_{j=1}^{N} (R_{l_j}(r_j, r_{j-1})) \prod_{j=1}^{N-1} r_j \, \mathrm{d}r_j \tag{25}$$

is the radial propagator of the l wave. Since K has to be unitary, the normalization factor is given by

$$B_N = \left(\frac{1}{2\pi \mathrm{i}\epsilon}\right)^N. \tag{26}$$

3. Evaluation of the radial propagator

It now remains to evaluate the radial propagator of equation (25), which may be written as

$$K_l(r'',r',t) = \lim_{N\to\infty} (-\mathrm{i}\alpha)^N \exp\left(\frac{\mathrm{i}\alpha}{2}(r'^2 + r''^2)\right)$$
$$\times \int \exp\{\mathrm{i}\beta(r_1^2 + r_2^2 + \ldots + r_{N-1}^2)\}$$
$$\times I_\mu(-\mathrm{i}\alpha r_0 r_1) \ldots I_\mu(-\mathrm{i}\alpha r_{N-1} r_N) \prod_{j=1}^{N-1} r_j \, \mathrm{d}r_j \tag{27}$$

where

$$\alpha = \epsilon^{-1} \qquad \beta = \alpha(1 - \tfrac{3}{4}\omega^2\epsilon^2) \qquad \mu = l + a + \tfrac{1}{2}. \tag{28}$$

The integrations may be performed by repeated use of the formula (Peak and Inomata 1969)

$$\int_0^\infty \exp(\mathrm{i}\alpha r^2) I_\nu(-\mathrm{i}br) I_\nu(-\mathrm{i}cr) r \,\mathrm{d}r = \frac{\mathrm{i}}{2\alpha} \exp\left(-\frac{\mathrm{i}(b^2+c^2)}{4\alpha}\right) I_\nu\left(-\frac{\mathrm{i}bc}{2\alpha}\right) \tag{29}$$

which is valid for $\mathrm{Re}\,(\nu) > -1$ and $\mathrm{Re}\,(\alpha) > 0$. The result is given by

$$K_l(r'', r', t) = \lim_{N\to\infty} \left(\frac{1}{\mathrm{i}}\right) \alpha_N \exp(\mathrm{i}p_N r'^2 + \mathrm{i}q_N r''^2) I_\mu(-\mathrm{i}\alpha_N r' r'') \tag{30}$$

where

$$\alpha_N = \alpha \prod_{k=1}^{N-1} \left(\frac{\alpha}{2\beta_k}\right) \tag{31}$$

$$p_N = \frac{\alpha}{2} - \sum_{k=1}^{N-1} \frac{\alpha_k^2}{4\beta_k} \tag{32}$$

$$q_N = \frac{\alpha}{2} - \frac{\alpha^2}{4\beta_{N-1}} \tag{33}$$

$$\alpha_1 = \alpha \qquad \alpha_{k+1} = \alpha \prod_{j=1}^{k} \left(\frac{\alpha}{2\beta_j}\right) \qquad k \geqslant 1 \tag{34}$$

$$\beta_1 = \beta \qquad \beta_{k+1} = \beta - \frac{\alpha^2}{4\beta_k} \qquad k \geqslant 1. \tag{35}$$

Using the method of Peak and Inomata (1969) it is easy to show that

$$\lim_{N\to\infty} \alpha_N = \Omega \operatorname{cosec} \Omega t \tag{36}$$

$$\lim_{N\to\infty} p_N = \frac{\Omega}{2} \cot \Omega t \tag{37}$$

$$\lim_{N\to\infty} q_N = \frac{\Omega}{2} \cot \Omega t \tag{38}$$

where

$$\Omega = \sqrt{\tfrac{3}{2}}\,\omega. \tag{39}$$

Thus, the radial propagator finally reads as

$$K_l(r'', r', t) = \left(\frac{\Omega}{\mathrm{i} \sin \Omega t}\right) \exp\left(\frac{\mathrm{i}\Omega}{2}(r'^2 + r''^2) \cot \Omega t\right) I_{l+a+1/2}\left(\frac{\Omega r' r''}{\mathrm{i} \sin \Omega t}\right). \tag{40}$$

4. Schrödinger equation and the propagator

We now show that the propagator can be expanded in terms of the eigenfunctions of the Schrödinger equation corresponding to the problem.

The Schrödinger equation for the problem, obtained after eliminating the CM motion and a subsequent transformation to 'plane polar' coordinates (r, θ), reads as

$$-\tfrac{1}{2}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\psi(r,\theta)+(\tfrac{1}{2}\Omega^2 r^2+U(r,\theta))\psi(r,\theta) = E\psi(r,\theta) \tag{41}$$

where $U(r, \theta)$ is the 'centrifugal potential' given by

$$U(r,\theta) = \frac{g}{2r^2 \sin^2\theta}. \tag{42}$$

The problem is separable in r and θ and one obtains the following eigenfunctions and energy values:

$$\psi_{nl}(r,\theta) = N_{nl} r^{l+a+1/2}\exp(-\tfrac{1}{2}\Omega r^2) L_n^{l+a+1/2}(\Omega r^2) \times (\sin\theta)^{a+1/2} C_l^{a+1/2}(\cos\theta) \qquad (0 \leqslant \theta \leqslant \pi) \tag{43}$$

$$E_{nl} = (2n+l+a+\tfrac{3}{2})\Omega \qquad n, l = 0, 1, 2, \ldots. \tag{44}$$

Here, C_l^{λ} and L_n^b are the Gegenbauer and Laguerre polynomials (Erdélyi 1953 pp174 and 188) and Ω is defined by equation (39).

On the other hand, as shown in § 2, the path integral for the problem also separates into the angular and radial parts. This is equivalent to the separation of the symmetry of the problem arising from the special nature of the potential $U(r, \theta)$. When evaluated, the radial part takes the closed form of equation (40). It is now possible to expand K_l of equation (40) in terms of Laguerre polynomials using the Hille–Hardy formula (Erdélyi 1953 equation (20) p189), to obtain

$$K_l(r'', r', t) = \sum_{n=0}^{\infty} \frac{n!2(\Omega)^{l+a+3/2}}{\Gamma(n+a+l+\frac{3}{2})} \exp\{-\mathrm{i}(2n+l+a+\tfrac{3}{2})\Omega t\} \times (r''r')^{l+a+1/2}\exp\{-\tfrac{1}{2}\Omega(r''^2+r'^2)\} L_n^{l+a+1/2}(\Omega r''^2) L_n^{l+a+1/2}(\Omega r'^2). \tag{45}$$

Finally, using equation (45) in equation (24), it is easily verified that the propagator

$$K(r'', \theta''; r', \theta', t) = \sum_{n,l=0}^{\infty} \psi_{nl}(r'', \theta'')\psi_{nl}(r', \theta')\exp(-\mathrm{i}E_{nl}t) \tag{46}$$

with $\psi_{nl}(r, \theta)$ and E_{nl} defined as in equations (43) and (44).

5. Conclusions

In this paper a one dimensional three body problem has been considered from the path integral point of view. The problem essentially involves the path integration of a system with the Lagrangian given in equations (12*a*–*c*). This Lagrangian has the interesting feature that the potential energy is the sum of a harmonic term and a 'centrifugal' term. The path integral, however, has been shown to be separable into a radial and an angular part. Physically this natural separation of symmetry from the propagator is related to the conservation of the quantity $\{p_\theta^2+(g/\sin^2\theta)\}$ where p_θ is the 'angular momentum'. Finally, the evaluation of the radial propagator in a closed form becomes possible as it involves only the harmonic part of the potential.

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